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A real-space renormalisation group study of a semi-infinite Ising model†

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Abstract. A modified Migdal-Kadanoff renormalisation group scheme, called the cluster-decimation approximation, is generalised and applied to study surface critical properties of the Ising model with a free surface. We determine the fixed points and calculate surface critical exponents of this model. Comparisons of our results with other real-space renormalisation group estimates are given.

1. Introduction

The semi-infinite Ising model is the simplest model for studying surface critical properties of magnetic systems. It has been studied by various methods, such as the mean-field approximation, the high-temperature series expansion method, Monte Carlo simulations and various real-space renormalisation group (RG) schemes (Binder 1983). This model displays complex and interesting critical behaviours. For the three-dimensional semi-infinite Ising ferromagnet, there are four kinds of phase transitions: the ordinary phase transition, the surface phase transition, the extraordinary phase transition and a surface-bulk phase transition. In this paper we investigate this model by using a modified Migdal-Kadanoff method, called the cluster-decimation (CD) approximation. This method was studied for bulk systems by Goldstein and Walker (1985) and Chen *et al* (1986). It retains the mathematical simplicity of the Migdal-Kadanoff (MK) method (Migdal 1976, Kadanoff 1976), and gives much better results than the MK method.

In the CD approximation the RG transformation of the coupling constant K is determined by the condition that the free energy for a small cluster of spins (with certain boundary conditions) is preserved exactly in the transformation. For the Ising model with a free surface, there are the bulk coupling constant K and surface coupling constant K_s. The generalisation of the CD approximation to systems with a surface is by no means trivial. We propose a cluster-decimation scheme for clusters on the surface, and obtain the RG transformation for K_s. In § 2 we describe the MK method for the semi-infinite system briefly, and give the RG transformation for any lattice dimensionality d and for an arbitrary rescaling factor b. In § 3 the cluster-decimation schemes for bulk clusters and for surface clusters are derived for $d=3$ and for $b=2$. From the RG transformation the fixed points and surface critical exponents of the three-dimensional semi-infinite Ising model are determined numerically in § 4. Critical

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properties calculated by the present method are compared with those determined by other real-space RG calculations. A summary and discussions are given in § 5.

2. MK approximation for the semi-infinite Ising model

The Hamiltonian (with the factor $-1/k_BT$) of the semi-infinite Ising model on a d-dimensional hypercubic lattice can be described as

$$H = \sum_{\langle ij\rangle_b} [Ks_is_j + h(s_i+s_j)/2d] + \sum_{\langle ij\rangle_{sb}} (Ks_is_j + hs_j/2d) + \sum_{\langle ij\rangle_s} [K_s s_i s_j + h(s_i+s_j)/(2d-2)] \tag{1}$$

where each spin s_i takes the values ± 1, $\langle ij\rangle_b$ indicates that the summation is over all nearest-neighbour (NN) pairs of spins s_i and s_j which are both in the bulk, $\langle ij\rangle_{sb}$ over all NN pairs of spins s_i and s_j with s_i on the surface and s_j in the bulk, and $\langle ij\rangle_s$ over all NN pairs of spins s_i and s_j with both spins on the surface. The NN couplings are K_s on the surface and K elsewhere. Figure 1(a) displays a three-dimensional semi-infinite Ising model. Each surface bond (denoted by a thick line) in the figure represents a coupling K_s and each bulk bond (a thin line) represents a coupling K. The magnetic fields are h_s on the surface and h in the bulk.

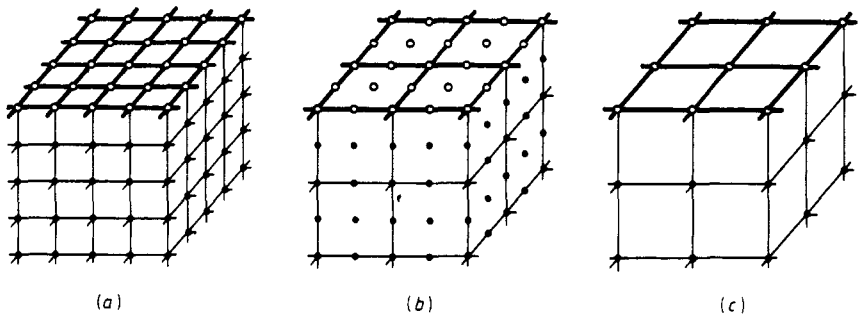

Figure 1. Bond moving and site decimation of the MK method (and the present method) with $b=2$, and $d=3$ for the semi-infinite Ising model. The original lattice (a) has surface couplings K_s (denoted by thick lines) and bulk couplings K (thin lines). The magnetic field is h_s on the surface and h in the bulk. The restructured lattice (b) has surface couplings $\tilde{K}_s$ (thick lines) and bulk couplings $\tilde{K}$ (thin lines). The magnetic field at a surface site connected by n surface bonds is $n\tilde{h}_s/(2d-2)$. The field at a bulk site connected by n bonds is $n\tilde{h}/2d$. The rescaled lattice (c) has surface couplings K_s' (thick lines) and bulk couplings K' (thin lines). The magnetic field is h_s' on the surface and h' in the bulk.

The MK method and its modifications consist of two operations: bond moving and site decimation. In the bond-moving step the system is divided into blocks, and all bonds are first moved into the faces and finally onto the edges of the blocks. We use the scheme that both the single-spin and the spin-spin interaction terms are moved together. That is, each bond contains a pair interaction and a field term as implied by (1). Figure 1(b) shows the restructured lattice obtained by bond-moving with $b=2$. In this lattice each surface bond (a thick line) carries a coupling $\tilde{K}_s$ and each bulk bond (a thin line) carries a coupling $\tilde{K}$. At a surface site connected by n surface bonds

there is a magnetic field $n\tilde{h}_s/(2d-2)$ (bulk bonds do not contribute magnetic fields to surface sites) and at a bulk site connected by n bonds, there is a magnetic field $n\tilde{h}/2d$.

The couplings $\tilde{K}$ and $\tilde{K}_s$ are functions of K and K_s, and the fields $\tilde{h}$ and $\tilde{h}_s$ are functions of K, K_s, h and h_s. In the simple MK method it is assumed that bond strengths are preserved in the bond-moving operation. Therefore

$$\tilde{K} = b^{d-1}K \tag{2a}$$

$$\tilde{K}_s = b^{d-2}[K_s + (b-1)K/2] \tag{2b}$$

and

$$\tilde{h} = b^{d-1}h \tag{3a}$$

$$\tilde{h}_s = b^{d-2}[h_s + (b-1)(d-1)h/2d]. \tag{3b}$$

In the restructured lattice, figure 1(b), isolated spins and spins connected by only two bonds are then decimated to obtain the rescaled lattice, figure 1(c). For the rescaled lattice the surface and the bulk coupling constants are K_s' and K', respectively, and the surface and the bulk magnetic fields are h_s' and h', respectively. It is easy to show that

$$K' = \tanh^{-1}(\tanh \tilde{K})^b \tag{4a}$$

$$K_s' = \tanh^{-1}(\tanh \tilde{K}_s)^b \tag{4b}$$

and

$$h' = \tilde{h}\Lambda^{(b)}(\tilde{K}) \tag{5a}$$

$$h_s' = \tilde{h}_s\Lambda^{(b)}(\tilde{K}_s) + (\tilde{h}/2d)[\Lambda^{(b)}(\tilde{K}) - 1] \tag{5b}$$

where the function $\Lambda^{(b)}$ (Burkhardt 1982) is given by

$$\Lambda^{(b)}(K) = \frac{1+\tanh K}{1-\tanh K}\,\frac{1-(\tanh K)^b}{1+(\tanh K)^b}. \tag{6}$$

Equations (2) and (4) describe the MK renormalisation transformation for the coupling constants, and (3) and (5) give the transformation for the magnetic fields. For $d=3$ and $b=2$, fixed points of the model are shown in table 1. These results are less accurate, as expected, because (2) and (3) are obtained by the approximation that bond strengths are preserved in the bond-moving step.

Table 1. Non-trivial fixed points (K^*, K_s^*) of the semi-infinite Ising model obtained by the present method with $x=2$, $y=1.5$ and $x=y=2$, respectively. Best estimates and results obtained by the MK method using $b=2$ and $b\to 1$, respectively, are given for comparison.

	Ordinary	Surface	Surface-bulk	Extraordinary
MK ($b=2$)	(0.065, 0.006)	(0, 0.303)	(0.065, 0.218)	(0.065, ∞)
MK ($b\to 1$)	(0.140, 0.026)	(0, 0.441)	(0.140, 0.346)	(0.140, ∞)
$x=2$, $y=1.5$	(0.215, 0.049)	(0, 0.492)	(0.215, 0.400)	(0.215, ∞)
$x=2$, $y=2$	(0.215, 0.048)	(0, 0.492)	(0.215, 0.384)	(0.215, ∞)
Best estimate	(0.222, —)	(0, 0.441)	(0.222, 0.355)	(0.222, ∞)

3. Modification of the MK method

In the simple MK scheme the coupling constants of the restructured lattice are proportional to those of the original lattice and the internal energy of the system is preserved only at $T=0$. In the CD approximation the coupling constants of the restructured lattice are so chosen that the internal energy is preserved both at $T=0$ and at high temperatures. It is difficult to give the general formulation of the CD approximation for arbitrary values of d and b. We only consider the three-dimensional lattice with $b=2$. Assume that a simple cubic lattice has N_s surface spins and N_b bulk spins (N_s and $N_b \gg 1$). It can be shown that the number of surface bonds L_s and the number of bulk bonds L_b of the lattice are

$$L_s = 2N_s \tag{7a}$$

and

$$L_b = 3N_b + N_s/2 \tag{7b}$$

respectively, and the number of surface bonds $\tilde{L}_s$ and the number of bulk bonds $\tilde{L}_b$ for the restructured lattice are, respectively,

$$\tilde{L}_s = N_s \tag{8a}$$

and

$$\tilde{L}_b = 3N_b/4 - N_s/8. \tag{8b}$$

The preservation of the energy at $T=0$ requires that

$$L_s K_s + L_b K = \tilde{L}_s \tilde{K}_s + \tilde{L}_b \tilde{K}. \tag{9}$$

Substituting (7) and (8) into (9), and comparing coefficients of N_b and N_s, we obtain

$$\tilde{K} = 4K \tag{10a}$$

and

$$\tilde{K}_s = 2K_s + K \qquad \text{for } T=0. \tag{10b}$$

Equations (10) are the same as (2) for $b=2$ and $d=3$.

The preservation of the internal energy for the semi-infinite system at high temperatures requires that

$$-k_B T(L_s K_s^2 + L_b K^2) = -k_B T(\tilde{L}_s \tilde{K}_s^2 + \tilde{L}_b \tilde{K}^2). \tag{11}$$

Equation (11) is obtained by deriving the leading terms of the high-temperature series expansions of the internal energy for the simple cubic lattice. Substituting (7) and (8) into (11), and comparing coefficients of N_b and N_s, respectively, we obtain

$$\tilde{K}^2 = 4K^2 \tag{12a}$$

and

$$\tilde{K}_s^2 = 2K_s^2 + K^2 \qquad \text{for} \qquad K, K_s \ll 1. \tag{12b}$$

We note that for any forms of $\tilde{K}$ and $\tilde{K}_s$ the free energy F of the system is preserved at high enough temperatures because the leading term of F at high temperatures, $-k_B T(N_s + N_b) \ln 2$, is independent of the coupling constants. For a bulk system, the transformation $\tilde{K}(K)$ of the CD approximation (Chen *et al* 1986) satisfies (10*a*) and (12*a*). In the following we will derive a transformation for the surface coupling $\tilde{K}_s(K, K_s)$ which satisfies (10*b*) and (12*b*).

Consider a cluster of eight spins and twelve bonds on the surface of the semi-infinite system as shown in the lower part of figure 2(*a*). We define the partition function of the cluster as

$$Z_3 = \sum_{s_1,\ldots,s_8} \exp[xK_s(s_1s_2+s_2s_3+s_3s_4+s_4s_1) + yK(s_1s_5+s_2s_6+s_3s_7+s_4s_8) + xK(s_5s_6+s_6s_7+s_7s_8+s_8s_5) + h_s(s_1+s_2+s_3+s_4)+h(s_5+s_6+s_7+s_8)]. \tag{13}$$

The parameters x and y describe the boundary condition, i.e. the interactions between the cluster and its neighbouring spins (Chen *et al* 1986). For free boundary conditions $x=y=1$. With conventional periodic boundary conditions $x=2$ and y can be taken as 1.5 because there are no vertical bonds connected to the cluster from above. In general x and y can be treated as independent parameters.

For the restructured lattice (with $b=2$), a cluster of eight spins contains only three bonds, as shown in the lower part of figure 2(*b*). The partition function of the surface cluster (with the same boundary condition as figure 2(*a*)) is

$$\tilde{Z}_3 = \sum_{s_1,\ldots,s_8} \exp[x\tilde{K}_s(s_1s_2+s_1s_4)+y\tilde{K}s_1s_5+\tilde{h}_s(ts_1+s_2+s_4)/t+\tilde{h}s_5/3] \tag{14}$$

where x and y are the same as those in (13), and $t=d-1=2$.

In (13) and (14), if we let $K_s=K$, $h_s=h$, $x=y$, $\tilde{K}_s=\tilde{K}$, $\tilde{h}_s=\tilde{h}$ and $t=d=3$, then Z_3 and $\tilde{Z}_3$ are the partition functions for bulk clusters in the simple cubic lattice and in the restructured lattice, respectively. In the CD approximation for bulk systems the transformations of the coupling constant $\tilde{K}(K)$ and the field $\tilde{h}(K,h)$ are given implicitly by the condition that, in the low-field limit, $Z_3(K,h)=\tilde{Z}_3(\tilde{K},\tilde{h})$ (with $K_s=K$, $h_s=h$, etc). If the partition functions are rewritten as

$$Z_3(K,h) = Z_3^{(0)}(K)+h^2Z_3^{(2)}(K)+\ldots \tag{15a}$$

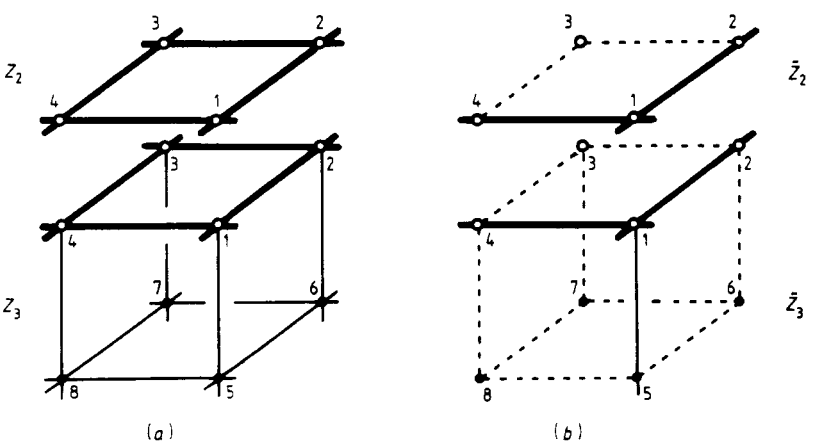

Figure 2. A 3D cluster contains eight spins (labelled $1,2,\ldots,8$). The partition functions of the 3D cluster (with a boundary condition) in the original lattice and in the restructured lattice are Z_3 and $\tilde{Z}_3$, respectively. A 2D cluster contains four surface spins (labelled 1, 2, 3 and 4). The partition functions of the 2D cluster (with boundary condition) in the original and in the restructured lattice are Z_2 and $\tilde{Z}_2$, respectively. The CD approximation requires that $\ln Z_2+\ln Z_3=\ln \tilde{Z}_2+\ln \tilde{Z}_3$.

and

$$\tilde{Z}_3(\tilde{K}, \tilde{h}) = \tilde{Z}_3^{(0)}(\tilde{K}) + \tilde{h}^2 \tilde{Z}_3^{(2)}(\tilde{K}) + \dots \tag{15b}$$

then $\tilde{K}(K)$ is defined implicitly by

$$Z_3^{(0)}(K) = \tilde{Z}_3^{(0)}(\tilde{K}) \tag{16}$$

and $\tilde{h}(K, h)$ is given through

$$h^2 Z_3^{(2)}(K) = \tilde{h}^2 \tilde{Z}_3^{(2)}(\tilde{K}). \tag{17}$$

Equation (16) reduces to (10*a*) and (12*a*), respectively, in appropriate limits.

For surface clusters the condition $Z_3(K, K_s) = \tilde{Z}_3(\tilde{K}, \tilde{K}_s)$ (with zero fields), however, does not yield a transformation $\tilde{K}_s(K, K_s)$ which satisfies (10*b*) and (12*b*). Physically this is because each spin in the bulk (s_5, s_6, s_7, s_8 in figure 2) is shared by eight clusters, while each spin on the surface (s_1, s_2, s_3, s_4 in figure 2) is shared only by four clusters. Also, each bond in the bulk is shared by four clusters, while each bond on the surface is shared only by two clusters. Therefore, $\ln Z_3 = \ln \tilde{Z}_3$ does not describe a proper transformation for $\tilde{K}_s$.

In addition to the three-dimensional (3D) clusters we can define two-dimensional (2D) clusters for surface spins. A 2D cluster contains four surface spins (s_1, s_2, s_3, s_4) as shown in figure 2. The partition functions of 2D clusters in the original and in the restructured lattices are given, respectively, by

$$Z_2 = \sum_{s_1, \dots, s_4} \exp[xK_s(s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1) + h_s(s_1 + s_2 + s_3 + s_4)] \tag{18a}$$

and

$$\tilde{Z}_2 = \sum_{s_1, \dots, s_4} \exp[x\tilde{K}_s(s_1 s_2 + s_1 s_4) + \tilde{h}_s(2s_1 + s_2 + s_4)/2]. \tag{18b}$$

We can consider each surface spin as shared by four 3D clusters and four 2D clusters, and consider each surface bond as shared by two 3D clusters and two 2D clusters. Then, the cluster decimation approximation for surface clusters is given by

$$\ln Z_2 + \ln Z_3 = \ln \tilde{Z}_2 + \ln \tilde{Z}_3. \tag{19}$$

The equality of the zeroth-order terms in the magnetic fields:

$$Z_2^{(0)}(K_s) Z_3^{(0)}(K, K_s) = \tilde{Z}_2^{(0)}(\tilde{K}_s) \tilde{Z}_3^{(0)}(\tilde{K}, \tilde{K}_s) \tag{20}$$

defines the transformation $\tilde{K}_s(K, K_s)$ implicitly. Equation (20) reduces to (10*b*) and (12*b*) for $T \to 0$ and $T \to \infty$, respectively, for any finite values of x and y.

From (19) the equality of the second-order terms in the magnetic fields (the first-order terms vanish) defines the transformation of $\tilde{h}_s$. The surface magnetic field $\tilde{h}_s$ depends on h as well as on h_s. As the surface magnetic exponents depend only on $\partial h_s'/\partial h_s$ (and are evaluated at zero fields), we can set $h = \tilde{h} = 0$ in (19). Then $\tilde{h}_s$ is determined implicitly by

$$h_s^2 (Z_3^{(0)} Z_2^{(2)} + Z_2^{(0)} Z_3^{(2)}) = \tilde{h}_s^2 (\tilde{Z}_3^{(0)} \tilde{Z}_2^{(2)} + \tilde{Z}_2^{(0)} \tilde{Z}_3^{(2)}) \tag{21}$$

where $Z_n^{(m)}$ are coefficients of h_s^m in Z_n, and $\tilde{Z}_n^{(m)}$ are coefficients of $\tilde{h}_s^m$ in $\tilde{Z}_n$ (with $h = \tilde{h} = 0$).

The site-decimation transformations of the CD approximation are the same as those for the simple MK method. They are given by (4) and (5) (with $b = 2$ and $d = 3$). Therefore, (16) and (4*a*) define the RG transformation $K'(K)$, whereas (20) and (4*b*)

define the transformation $K_s'(K, K_s)$. The RG transformation of the bulk magnetic field $h'(K, h)$ is given by (17) and (5*a*), and the transformation of the surface magnetic field $h_s'(K, K_s, h=0, h_s)$ is given by (21) and (5*b*). The explicit expressions of (16), (17), (20) and (21) are too lengthy to be presented here, but they can be derived easily.

4. Fixed points and surface critical exponents

There are seven fixed points for the three-dimensional semi-infinite Ising model (Burkhardt and Eisenriegler 1977, 1978). The various fixed points are: paramagnetic (P), bulk ferromagnetic (BF), surface ferromagnetic (SF), ordinary critical (O), surface critical (S), surface-bulk multicritical (SB) and extraordinary (E). The first three fixed points are trivial. The paramagnetic fixed point has $K^* = K_s^* = 0$. For the SF fixed point $K^* = 0$ and $K_s^* \to \infty$, and for the BF fixed point $K^* = K_s^* \to \infty$. Other fixed points are non-trivial. It is straightforward to determine these fixed points numerically from (4), (16) and (20). We choose $x = 2$ in our calculations as it has been shown (Chen *et al* 1986, Lee *et al* 1988) that the appropriate value of x to be used is 2 for three-dimensional lattices. We do not know in advance what value of y should be used. We try $y = 1.5$ and 2, respectively, in our calculations. Table 1 shows various non-trivial fixed points determined by the present method and by the MK method with $b = 2$ and $b \to 1$ (Lipowsky and Wagner 1981), respectively. The best known values (Binder and Hohenberg 1974) are also given in table 1 for comparison. We see that the present method obtains much better results than the simple MK approximation.

Bulk critical exponents y_t and y_h of the system are determined by the derivatives $\partial K'/\partial K$ and $\partial h'/\partial h$ (evaluated at $K = K^*$ and $h = 0$). They are unaffected by the existence of the surface and have been reported before (Chen *et al* 1986). The surface thermal exponent y_{t_1} and the surface field exponent y_{h_1} are given respectively by

$$y_{t_1} = \ln(\partial K_s'/\partial K_s)/\ln b \tag{22a}$$

and

$$y_{h_1} = \ln(\partial h_s'/\partial h_s)/\ln b \tag{22b}$$

where $b = 2$ and the derivatives are evaluated at the fixed point considered. Various y_{t_1} and y_{h_1} we obtained are listed in table 2, together with those obtained by different real-space renormalisations. Here CCL and CCL′ are results of the two-cell cluster approximation with the usual and a changed majority rules (Burkhardt and Eisenriegler 1977, Švrackić *et al* 1980), CUM and CUM′ are obtained by the lowest-order cumulant expansion with the usual and a changed majority rules (Dunfield and Noolandi 1980, Burkhardt and Eisenriegler 1977). These studies used three coupling constants in the Hamiltonian (this is why their fixed points are not shown in table 1). The results of Kadanoff's variational method (KV) are calculated by Burkhardt and Eisenriegler (1978) for the body-centred cubic lattice. Exact values (or best estimates) are from Binder and Hohenberg (1974). The present method obtains much better estimates of $y_{t_1}^{(S)}$, $y_{t_1}^{(SB)}$ and $y_{h_1}^{(O)}$ than the MK method ($b \to 1$ or $=2$). Our estimates of $y_{h_1}^{(S)}$ and $y_{h_1}^{(SB)}$ are not as good as the MK results (but the differences are only 2%). In general, our results are also better than those determined by the cluster approximation and by the cumulant expansion method. From bulk exponents y_t, y_h and the surface exponents shown in table 2, other critical exponents of the semi-infinite system can be obtained from scaling laws (Binder and Hohenberg 1972, 1974, Wortis and Švrakić 1982, Švrakić and Wortis 1977, Burkhardt and Eisenriegler 1977).

Table 2. Surface critical exponents y_{t_1}, y_{h_1} and critical surface enhancement D_c of the semi-infinite Ising model obtained by the present method ($x = 2, y = 1.5$ and $x = 2, y = 2$) and by other real-space renormalisation schemes. The abbreviations of the methods are explained in the text.

	$y_{t_1}^{(S)}$	$y_{t_1}^{(SB)}$	$y_{h_1}^{(O)}$	$y_{h_1}^{(S)}$	$y_{h_1}^{(SB)}$	$y_{h_1}^{(E)}$	D_c
CCL	0.46	0.08	1.13	1.84	1.21	1.98	0.11
CCL′	0.89	0.72	0.79	1.66	1.56	2	0.36
CUM	1.12	0.05	1.54	2.65	1.62	∞	—
CUM′	1.01	0.79	0.56	2.15	1.97	2	0.31
KV	1.00	0.87	0.74	1.88	1.75	2	—
MK ($b = 2$)	0.74	0.61	1.20	1.88	1.82	2	2.36
MK ($b \to 1$)	0.75	0.63	1.19	1.88	1.82	2	1.48
$x = 2, y = 1.5$	0.84	0.71	1.05	1.92	1.86	2	0.86
$x = 2, y = 2$	0.84	0.70	1.07	1.92	1.86	2	0.79
Best estimate	1	—	1.0 ± 0.2	1.875	—	2	0.6 ± 0.1

The surface enhancement of the semi-infinite system is defined as $D = (K_s - K)/K$. The surface enhancement at the SB fixed point is called the critical surface enhancement D_c. For $D > D_c$ the system exhibits two phase transitions: a surface phase transition and an extraordinary phase transition. When $D < D_c$ there is only an ordinary phase transition. The critical surface enhancement obtained by the present method and other real-space renormalisations are also included in table 2. We see that the value of D_c determined by the present method (either $y = 1.5$ or 2) is much better than the estimates of other real-space renormalisation group schemes.

5. Summary and discussion

We have proposed a modification of the Migdal-Kadanoff method for the semi-infinite Ising model which contains the surface coupling constant K_s and the bulk coupling constant K. The renormalisation transformation of the coupling K is just the same as that of the CD approximation in which the free energy (and hence other thermodynamic quantities) for a cluster of eight spins in the bulk is preserved exactly. Similar to the CD approximation for bulk systems the transformation of the surface coupling K_s is defined through the preservation of the free energy for a cluster on the surface. The free energy of a surface cluster is taken to be the sum of free energies of a 3D cluster of eight spins and a 2D cluster of four spins on the surface as shown in figure 2. Two parameters x and y are used to describe the boundary condition of the cluster considered.

In the simple MK method the coupling constants of the restructured lattice, $\tilde{K}$ and $\tilde{K}_s$, are linear combinations of K and K_s, such that the internal energy of the whole lattice is preserved at $T = 0$. The internal energy, however, is not preserved at high temperatures. In the present method $\tilde{K}$ and $\tilde{K}_s$ are functions of K and K_s defined by (16) and (20). Therefore, the internal energy of the whole system is preserved both at $T = 0$ and at high temperatures (to first order in T^{-1}) for all values of x and y.

For the parameter x we use the periodic boundary condition, $x = 2$, as suggested by the studies of bulk systems (Chen *et al* 1986, Lee *et al* 1988). We have considered two values of y—1.5 and 2—in this calculation. The differences between the estimates

of fixed points and surface critical exponents obtained by these two values of y are within a few per cent. The present method is mathematically simpler than other real-space renormalisation schemes such as the finite cluster approximation, the cumulant expansion and Kadanoff's variational transformation. Critical properties of the semi-infinite Ising model obtained by the present method are in general better than the results of the finite cluster approximation and the cumulant expansion. In particular the present method yields a very good estimate of the critical surface enhancement D_c.

In the present study the magnetic-field terms are moved together with the Ising interactions in the bond-moving operation. We have also considered another bond-moving scheme: that the field terms are not moved with the pair-interaction terms. In this case the field exponents obtained are much poorer.

As mentioned in § 4 some studies of the semi-infinite systems considered three coupling constants. The coupling between a surface spin and its NN spin in the bulk is $K_\perp$. That is, the coupling constant in the second term of (1) is $K_\perp$ instead of K. For this model we need one more equation besides (16) and (20) to determine $\tilde{K}$, $\tilde{K}_\perp$ and $\tilde{K}_s$. A simple and reasonable assumption is that $\tilde{K}_\perp/\tilde{K} = (K_\perp/K)^n$ for $n = 1$ or any positive value. Under this assumption the subspace $K_\perp = K$ is invariant under the renormalisation transformation and all fixed points lie in the two-dimensional coupling constant subspace. This property is also observed in the simple MK method (Lipowsky and Wagner 1981). Therefore, when we consider the semi-infinite Ising model with three coupling constants, we obtain the same fixed points (with $K_\perp^* = K^*$) as shown in table 1.

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